MOTION OF A LIQUID IN A FRACTURED POROUS STRATUM WITH UNSTEADY-STATE FILTRATION

S. N. Bagir-zade and G. P. Guseinov

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Papers [1-4] are devoted to investigation of the motion of a homogeneous liquid in a fractured porous stratum with unsteady-state filtration. In [3] a study was made of the motion of a homogeneous liquid toward a central borehole from a round fractured porous stratum of infinite extension, and the conclusion was drawn that the curve of the re-establishment (lowering) of the face pressure of the borehole after shutdown (start-up) of the well is of a "two-layer" character. Paper [4] is devoted to a substantiation of such a change in the re-establishment (lowering) of the face pressure of the borehole. However, both in [3] and in [4] only approximate formulas describing this process are obtained. The present paper gives exact solutions of the problems of the unsteady-state filtration of a homogeneous liquid toward a central borehole from round fractured porous strata in two characteristic cases: when the medium has an impermeable external boundary, and when the medium is unbounded in extension. The article presents numerical calculations which confirm the deduction that the curve of the re-establishment (lowering) of the face pressure of the borehole has a "two-layer" character of the change in the case where the fractured porous stratum is unbounded in extension. In the case of a closed fractured porous stratum, the curve for the change with time of the face pressure of the borehole is always greater than that for a granular medium. In the solution of the problems it is assumed that the output of the borehole, which is ideal in character with respect to the degree of opening, is constant over the course of the whole process of exploitation.

In accordance with the theory of the motion of a homogeneous liquid in a fractured porous medium proposed in [1-3], solution of the problem posed reduces to integration of the system of equations

$$\frac{\partial^2\psi_2^{(i)}}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\psi_2^{(i)}}{\partial\xi}-(1-\omega)\frac{\partial\psi_1^{(i)}}{\partial\tau}=\omega\frac{\partial\psi_2^{(i)}}{\partial\tau} \tag{1}$$

$$(1-\omega)\frac{\partial\psi_1^{(i)}}{\partial\tau}=\lambda\left(\psi_2^{(i)}-\psi_1^{(i)}\right)\qquad(i=1,2)$$

with the following initial and boundary conditions:

$$\psi_1^{(i)}(\xi,0)=\psi_2^{(i)}(\xi,0)=0\qquad(i=1,2) \tag{2}$$

$$\left(\xi\frac{\partial\psi_2^{(i)}}{\partial\xi}\right)_{\xi=1}=-1\qquad(i=1,2) \tag{3}$$

$$\left(\frac{\partial\psi_2^{(2)}}{\partial\xi}\right)_{\xi=R}=0 \tag{4}$$

$$\psi_2^{(1)}(\infty,\tau)=0 \tag{5}$$

Here the following notation is adopted:

$$\psi_j^{(i)}(\xi,\tau)=2\pi\frac{k_2h}{q\mu}\left[p_0-p_j^{\,i}(\xi,\tau)\right]\qquad(i,j=1,2)$$

$$\omega=\frac{m_2\beta_2}{m_1\beta_1+m_2\beta_2},\qquad \tau=\frac{k_2t}{\mu r_b^{\,2}(m_1\beta_1+m_2\beta_2)} \tag{6}$$

$$\lambda=\alpha r_b^{\,2}\frac{k_1}{k_2},\qquad \xi=\frac{r}{r_b},\qquad R=\frac{r_k}{r_b}$$

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where α is a parameter of the fractured porous medium, characterizing the exchange of liquid between slightly permeable blocks and fractures; p_0 and $p(\xi, \tau)$ are the initial and simultaneous pressures, respectively; m is the porosity; k and β are the coefficients of permeability and elastic capacity; μ is the viscosity; t is the time; r, r_b, and r_k are the instantaneous radius, the radius of the borehole, and the radius of the impermeable external boundary of the stratum. The superscripts 1 and 2 in the functions of the pressure correspond to media which are unbounded in extension and closed; and the subscripts correspond to the systems of blocks and fractures of the medium.

Applying a Laplace transform with respect to the time to systems (1) and boundary conditions (3)-(5), and then eliminating the function for the lowering of the pressure in the system of blocks from the system obtained, we can write

$$\frac{d^2\Psi_2^{(i)}}{d\xi^2} + \frac{1}{\xi}\frac{d\Psi_2^{(i)}}{d\xi} - S_*\Psi_2^{(i)}(\xi, s) = 0 \qquad (i = 1, 2) \tag{7}$$

where

$$S_* = s[\omega(1-\omega)s + \lambda][(1-\omega)s + \lambda]^{-1} \tag{8}$$

s is the parameter of the Laplace transform, and $\Psi(\xi, s)$ is a Laplace transform of the function $\Psi(\xi, \tau)$.

The general solution of Eqs. (7) has the form [5, 6]

$$\Psi_2^{(i)}(\xi, s) = AI_0(\xi\sqrt{S_*}) + BK_0(\xi\sqrt{S_*}) \qquad (i = 1, 2) \tag{9}$$

Here $I_0(x)$ and $K_0(x)$ are Bessel functions of zero order with an imaginary argument of the first and second kinds, respectively.

Finding the integration constants A and B from boundary conditions (3) and (5) for an infinite stratum and (3), (4) for a closed stratum, to which a Laplace transform has previously been applied, the solutions of the problem can be represented in the form

$$\Psi_2^{(1)}(\xi, s) = K_0(\xi\sqrt{S_*})[s\sqrt{S_*}K_1(\sqrt{S_*})]^{-1} \tag{10}$$

$$\Psi_2^{(2)}(\xi, s) = \frac{1}{s\sqrt{S_*}}\frac{I_0(\xi\sqrt{S_*})K_1(R\sqrt{S_*}) + I_1(R\sqrt{S_*})K_0(\xi\sqrt{S_*})}{I_1(R\sqrt{S_*})K_1(\sqrt{S_*}) - I_1(\sqrt{S_*})K_1(R\sqrt{S_*})} \tag{11}$$

Using the relationship from [5]

$$s[\Psi_2^{(2)}(\xi, s)]_{S_*=s} = \frac{2}{R^2-1} - \pi\sum_{n=1}^{\infty}\beta_n U_n(\xi, R)\exp(-\beta_n^2\tau) \tag{12}$$

where

$$U_n(\xi, R) = J_1(\beta_n R)\frac{J_1(\beta_n)Y_0(\beta_n\xi) - J_0(\beta_n\xi)Y_1(\beta_n)}{J_1^2(\beta_n R) - J_1^2(\beta_n)} \tag{13}$$

the inverse transform of (11), using the generalized Éfros multiplication theorem [7], may be found in the form

$$\begin{aligned}\psi_2^{(2)}(\xi, \tau) = \frac{2\tau}{\omega(R^2-1)} - \pi\sum_{n=1}^{\infty}\frac{U_n(\xi, R)}{\beta_n}\left[1 - \exp\left(-\frac{\beta_n^2\tau}{\omega}\right)\right] - \\ - \int_0^{\tau/\omega} g(\tau, \theta)\left[\frac{2}{R^2-1} - \pi\sum_{n=1}^{\infty}\beta_n U_n(\xi, R)\exp(-\beta_n^2\theta)\right]d\theta\end{aligned} \tag{14}$$

where

$$g(\tau, \theta) = \begin{cases}\lambda\exp\left(-\lambda\frac{\tau-\omega\theta}{1-\omega}\right)\int_0^{\theta} e^{-\lambda z}I_0\left[2\lambda\left(z\frac{\tau-\omega\theta}{1-\omega}\right)^{1/2}\right]dz & \text{at} \quad \theta < \frac{\tau}{\omega} \\ 0 \quad \text{at} \quad \theta > \tau/\omega\end{cases} \tag{15}$$

β_n are the roots of the equation

$$J_1(\beta_n R)Y_1(\beta_n) - J_1(\beta_n)Y_1(\beta_n R) = 0 \tag{16}$$

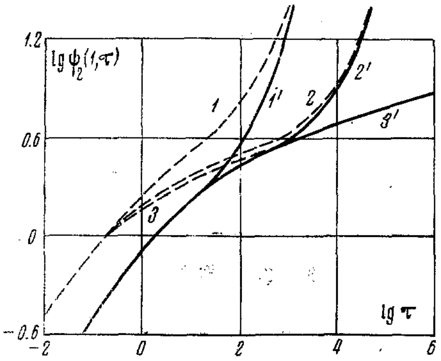

Fig. 1

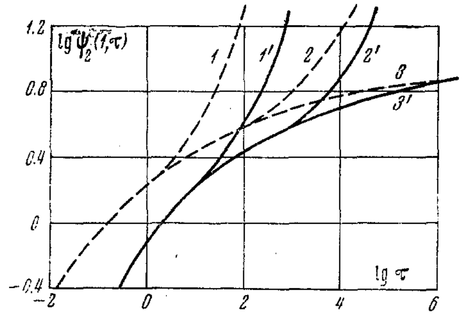

Fig. 2

$J_\nu(x)$ and $Y_\nu(x)$ ($\nu = 0.1$) are Bessel functions of a real argument, of the first and second order with respect to ν, respectively.

With $\xi = 1$, using the recurrence formula [8]

$$J_1(x)\,Y_0(x) - J_0(x)\,Y_1(x) = 2/\pi x \tag{17}$$

the expression for the lowering of the pressure at the wall of a borehole, for a fractured porous stratum with continuous removal of the liquid, can be represented in the form

$$\psi_2^{(2)}(1,\tau) = \frac{2\tau}{\omega(R^2-1)} - 2\sum_{n=1}^{\infty}\frac{U_n(1,R)}{\beta_n^2}\left[1-\exp\left(\frac{-\beta_n^2\tau}{\omega}\right)\right] - $$
$$-2\int_0^{\tau/\omega} g(\tau,\theta)\left[(R^2-1)^{-1} - \sum_{n=1}^{\infty} U_n(1,R)\exp(-\beta_n^2\theta)\right]d\theta \tag{18}$$

where

$$U_n(1,R) = J_1(\beta_n R)\,[J_1^2(\beta_n R) - J_1^2(\beta_n)]^{-1} \tag{19}$$

Formula (18) may be cast into form convenient for calculations if the corresponding formula for a granular medium is used [5, 6]

$$\psi_2^{(2)}(1,\tau) = \frac{0.5+2\tau}{R^2-1} + \frac{(4\ln R-3)R^4+2R^2+1}{4(R^2-1)^2} + 2\sum_{n=1}^{\infty}\beta_n^{-2}U_n(1,R)\exp(-\beta_n^2\tau) \tag{20}$$

In view of the fact that at the initial moment of time ($\tau = 0$) the lowering of the pressure at the face of the borehole must be equal to zero, from (20) we can set up the equality

$$\sum_{n=1}^{\infty}\beta_n^{-2}U_n(1,R) = \frac{0.25}{1-R^2} - \frac{(4\ln R-3)R^4+2R^2+1}{8(R^2-1)^2} \tag{21}$$

Taking account of (21), formula (18) is rewritten in the form

$$\psi_2^{(2)}(1,\tau) = \frac{0.25+2\tau/\omega}{R^2-1} + \frac{(4\ln R-3)R^4+2R^2+1}{8(R^2-1)^2} + $$
$$+2\sum_{n=1}^{\infty}\beta_n^{-2}U_n(1,R)\exp(-\beta_n^2\tau/\omega) - 2\int_0^{\tau/\omega} g(\tau,\theta)\left[\frac{1}{R^2-1} - \sum_{n=1}^{\infty}U_n(1,R)\exp(-\beta_n^2\theta)\right]d\theta \tag{22}$$

Use of formula (22), which is an exact solution of the problem for a closed stratum, for any given moment of time permits calculating the value of the lowering of the face pressure of a borehole when exploiting a deposit with fractured porous types of collectors and with continuous withdrawal of liquid.

It can be shown that with $\omega = 1$, the term containing $g(\tau, \theta)$ reverts to zero, and formula (22) assumes the form (20). The explanation of this is the fact that, with this value of the parameter of the fracture capacity ω, the starting system of equations also goes over into the equation of piezo-conductivity for a granular medium.

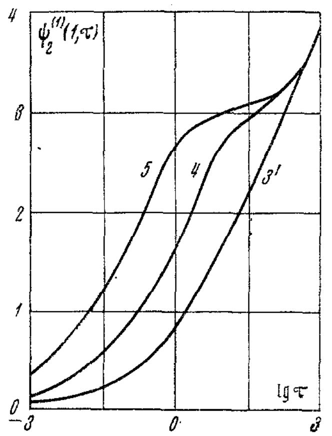

Fig. 3

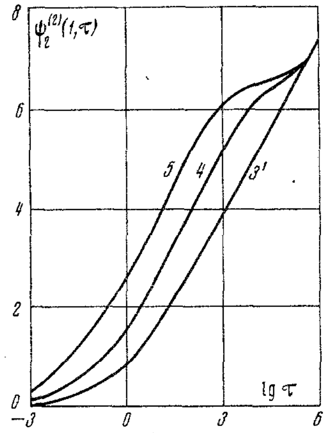

Fig. 4

Making the transition as before from the transform to the inverse transform, an accurate solution to the problem for an infinite fractured porous stratum can be found in the form

$$\psi_2^{(1)}(1,\tau)=\frac{4}{\pi^2}\int_0^\infty\frac{1-\exp(-u^2\tau/\omega)}{J_1^2(u)+Y_1^2(u)}\frac{du}{u^3}-\frac{4}{\pi^2}\int_0^{\tau/\omega}g(\tau,\theta)\,d\theta\int_0^\infty\frac{\exp(-u^2\theta)}{J_1^2(u)+Y_1^2(u)}\frac{du}{u}\tag{23}$$

If in (10) the parameter of the transform s is assumed to be sufficiently large, the quantity λ in the expression for S_* can be neglected and, at small values of the time, the formula for the lowering of the face pressure assumes the form

$$\psi_2^{(1)}(1,\tau)\approx\frac{4}{\pi^2}\int_0^\infty\frac{1-\exp(-u^2\tau/\omega)}{J_1^2(u)+Y_1^2(u)}\frac{du}{u^3}\tag{24}$$

Expression (24) differs from the exact formula for a granular medium only by the factor ω^{-1} with the exponent.

It must be noted that in [3] asymptotic solutions are obtained to the problems under consideration, which are applicable with sufficiently large values of the time. In this case, in formulas (10) and (11), the authors used approximate expressions for the Bessel functions of an imaginary argument, which facilitated the transition to the inverse transform. These solutions have the form [3]

$$\psi_2^{(1)}(1,\tau)\approx0.5\left\{\ln\tau+0.809+Ei\left[\frac{-\lambda\tau}{\omega(1-\omega)}\right]-Ei\left(\frac{-\lambda\tau}{1-\omega}\right)\right\}\tag{25}$$

$$\psi_2^{(2)}(1,\tau)\approx\frac{2}{R^2-1}\left\{\frac{1}{4}+\tau+\frac{(1-\omega)^2}{\lambda}\left[1-\exp\left(\frac{-\lambda\tau}{\omega(1-\omega)}\right)\right]\right\}$$
$$+\,0.25(R^2-1)^{-2}\left[(4\ln R-3)R^4+2R^2+1\right]\tag{26}$$

Ei(−x) is an integral exponential function [9].

With $\omega=1$ or $\lambda\to\infty$, these solutions go over into the corresponding formulas for granular media [5, 6]

$$\psi_2^{(1)}(1,\tau)\big|_{\omega=1}\approx0.5\ln\tau+0.4045\tag{27}$$

$$\psi_2^{(2)}(1,\tau)\big|_{\omega=1}\approx\frac{0.5+2\tau}{R^2-1}+\frac{(4\ln R-3)R^4+2R^2+1}{4(R^2-1)^2}\tag{28}$$

Formulas (20), (24), (27), and (28), expressing the dependence of the lowering of the face pressure on the time for granular media, are widely known in underground petroleum hydrodynamics, and have been tabulated over a sufficiently wide range of change in the time [5, 6]. Such dependences are plotted using solid lines on Figs. 1 and 2 for different deposits, for which the impermeable external contours are equal to 10 and 50 units of the dimensionless radius R. The dotted lines on these figures correspond to deposits with fractured porous types of collectors at the same values of the radius R. The calculations were made using formulas (22) and (26) at $\omega=0.1$ for both figures, and $\lambda=5\cdot10^{-3}$ and $5\cdot10^{-6}$ for Figs. 1 and 2, respectively.

Curves 3 and 3', being envelopes of the above-mentioned curves, describe the change with time of the face pressure for boreholes in deposits of inifinite extension, with fractured porous and granular types of collector, respectively. The segments of these curves after branching were calculated using formulas (23) and (25); the exact formulas (22) and (23), corresponding to closed and infinite strata, are in good agreement.

The results of calculations with different values of the parameters ω and λ for an infinite stratum are shown on Figs. 3 and 4 in semilogarithmic coordinates. The curves 3' on these figures correspond to the same deposits as on Figs. 1 and 2. Curves 4 and 5 differ by the value of the parameter ω equal to 0.1 and 0.01, respectively. The difference between Fig. 3 and 4 consists in values of the parameter λ equal to $5 \cdot 10^{-3}$ and $5 \cdot 10^{-6}$, respectively.

Curves 4 and 5 (Figs. 3 and 4) confirm the two-layer character of the change in the curves for the lowering of the face pressure in boreholes in a fractured porous stratum, with fixed values of the parameters ω and λ.

We note that, on the initial straight-line segment of curves 4 and 5 (Figs. 3 and 4) there is agreement between the exact and asymptotic formulas (23) and (25).

From an analysis of the curves of all the figures presented, the following conclusions may be drawn with respect to the motion of a liquid toward a central borehole in fracture porous strata, with unsteady-state filtration:

1) the lowering of the face pressure of a borehole in a fractured stratum is greater than or equal to the same value for a granular stratum, at an identical value of the time;

2) a decrease in the parameter of the fracture capacity ω lowers the face pressure and, with an increase in the time of exploitation, this lowering becomes equal to the lowering of the pressure in the case of a granular medium, when both media are unbounded in their extension. Under these circumstances, the value of the parameter λ, characterizing the degree of retardation of the transfer of liquid between the blocks and fractures of the medium, exerts a considerable effect on the equalization time (curves 3, 4, and 5);

3) the time dependence of the lowering of the free pressure of a borehole in an infinite stratum differs from the same value for a granular medium by the quantity

$$\psi_2^{(1)}(1,\tau) - \psi_2^{(1)}(1,\tau)\big|_{\omega=1} = 0.5Ei\left[\frac{-\lambda\tau}{\omega(1-\omega)}\right] - 0.5Ei\left(\frac{-\lambda\tau}{1-\omega}\right) \tag{29}$$

which reverts to zero with an increase in the exploitation time. In the case of a closed stratum, the following relationship holds:

$$\psi_2^{(2)}(1,\tau) - \psi_2^{(2)}(1,\tau)\big|_{\omega=1} = \frac{2(1-\omega)^2}{\lambda(R^2-1)}\left[1 - \exp\left(\frac{-\lambda\tau}{\omega(1-\omega)}\right)\right] \tag{30}$$

which, in distinction from the preceding, reverts to the constant

$$2(1-\omega)^2\lambda^{-1}(R^2-1)^{-1}$$

at those values of time where

$$\exp\left(\frac{-\lambda\tau}{\omega(1-\omega)}\right) = 0$$

In conclusion we note that the values of the functions entering into the calculating formulas were taken from [9-11].

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